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LETTER TO THE EDITOR

On a class of infinite-dimensional representations of $U_q(sl(2))$

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Abstract. A class of infinite-dimensional representations of the quantum universal enveloping algebra $U_q(sl(2))$ is considered. Equivalence and reducibility conditions are examined.

Quantum groups, or quasi-triangular Hopf algebras [1, 2], have attracted much attention recently. The main examples of these are q-deformations $U_q(\mathfrak{g})$ of universal enveloping algebras for any simple Lie algebra $\mathfrak{g}$ [3, 4] (see e.g. [1, 2, 5] for the general setting). The representation theory of $U_q(\mathfrak{g})$ has been considered in [6, 7] for values of the deformation parameter q which are not roots of unity, and, for example, in [8, 9] in the degenerate case $q^p = 1$ (the latter is more interesting both from a mathematical and physical point of view). In the simplest example of $U_q(sl(2))$ a detailed investigation has been carried out in [10] (and [11, 12] in the degenerate case). All these papers are devoted to the finite-dimensional representations.

We consider here a simple class of *infinite-dimensional* representations of the quantum universal enveloping algebra (QUEA) $U_q(sl(2))$ indexed by four real numbers $[\delta, \varepsilon; s, t]$ where δ is a substitute for the spin, s and t are some exponents and $\varepsilon \in \{0, \frac{1}{2}\}$. Similar representations in the classical $q = 1$ case were proposed by one of us [13] in 1968.

We define the QUEA $U_q(sl(2))$ as the complex unital associative algebra consisting of polynomials in $X^\pm$ and convergent power series in H where (for $q \in \mathbb{C}\backslash\{0, 1\}$)

$$[H, X^\pm] = \pm X^\pm \qquad [X^+, X^-] = \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}}. \tag{1}$$

The symbols $q^{\pm H}$ are usually considered as a generator; including H itself in $U_q(sl(2))$ allows the limit $q \to 1$ to be taken which reduces (1) to the defining relations of the Lie algebra $sl(2)$.

We shall not use the Hopf algebra structure of $U_q(sl(2))$ throughout this letter.

Quantum numbers $[x]_q$ and q-factorials $[n]_q!$ are defined by

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \qquad (x \in \mathbb{C}) \tag{2a}$$

$$[n]_q! = [n]_q [n-1]_q! \qquad [1]_q! = 1 \qquad (n \in \mathbb{N}). \tag{2b}$$

The central element

$$C = [H]_q [H+1]_q + X^- X^+ \tag{3}$$

is obviously a q-deformation of the classical Casimir operator.

We shall consider a set of representations of $U_q(sl(2))$ characterized by the following conditions. Let $\delta \in \mathbb{R}$, $\varepsilon \in \{0, \frac{1}{2}\}$ and:

(i) the representation space is a Hilbert space;

(ii) the operator representing H has a purely discrete simple spectrum consisting of the points $\{m \in \mathbb{R}, m - \varepsilon \in \mathbb{Z}\}$ and the corresponding eigenvectors form an orthogonal basis;

(iii) the operator representing C is proportional to the unit operator and (using the same notation for algebraically homomorphic objects)

$$Ce_m = [\delta]_q[\delta+1]_q e_m. \tag{4}$$

As $HX^{\pm}e_m = (m \pm 1)X^{\pm}e_m$, it follows from (ii) that $X^{\pm}e_m$ is proportional to $e_{m\pm 1}$ (or gives the zero vector, i.e. e_m is a highest resp. lowest weight vector). Let

$$X^{\pm}e_m = \alpha^{\pm}_{\delta,m} e_{m\pm 1}. \tag{5}$$

Then, from (3)-(5)

$$\alpha^{-}_{\delta,m+1}\alpha^{+}_{\delta,m} = [\delta - m]_q[\delta + m + 1]_q. \tag{6}$$

The last equation in (1) is automatically satisfied. Any solution of (6) provides a Hilbert space representation of $U_q(sl(2))$. We set

$$\begin{aligned} \alpha^{-}_{\delta,m} &= [\delta + m]^{s}_q[\delta - m + 1]^{1-t}_q \mu_{\delta,m-1} \\ \alpha^{+}_{\delta,m} &= [\delta - m]^{t}_q[\delta + m + 1]^{1-s}_q \mu^{-1}_{\delta,m} \end{aligned} \tag{7}$$

where s and t are, in general, arbitrary real numbers and $\mu_{\delta,m}$ are some non-zero complex constants. We take the principal branch of the multivalued functions and will restrict the range of s and t to $0 \leqslant s, t \leqslant 1$ whenever some of the q numbers in (7) can vanish.

Representations π_1 and π_2 for which there exists a bounded intertwiner A with a bounded inverse are equivalent. Let $X_i = \pi_i(X)$, $i = 1, 2$, $X \in U_q(sl(2))$. Then, from $AH_1 = H_2A$ where

$$\begin{aligned} H_1 e_n &= n e_n \qquad & n - \varepsilon_1 \in \mathbb{Z} \\ H_2 f_\nu &= \nu f_\nu \qquad & \nu - \varepsilon_2 \in \mathbb{Z} \end{aligned} \tag{8}$$

and

$$Ae_n = \sum_{\nu \in \mathbb{Z} - \varepsilon_2} a_{n\nu} f_\nu$$

it follows that $a_{n\nu} = a_n \delta_{n\nu}$. Boundedness of A and A^{-1} implies

$$\varepsilon_1 = \varepsilon_2 \qquad 0 < c_1 < |a_n| < c_2. \tag{9}$$

From $AC_1 = C_2A$ one obtains the condition

$$[\delta_1 - \delta_2]_q[\delta_1 + \delta_2 + 1]_q = 0. \tag{10}$$

To complete the list of relations among parameters of equivalent representations, one has to solve the equations $AX^{\pm}_1 = X^{\pm}_2 A$. These give, respectively:

$$a_{m+1}[\delta_1 - m]^{t_1}_q[\delta_1 + m + 1]^{1-s_1}_q \mu^{-1}_{\delta_1,m} = a_m[\delta_2 - m]^{t_2}_q[\delta_2 + m + 1]^{1-s_2}_q \mu^{-1}_{\delta_2,m} \tag{11a}$$

$$a_{m-1}[\delta_1 + m]^{s_1}_q[\delta_1 - m + 1]^{1-t_1}_q \mu_{\delta_1,m-1} = a_m[\delta_2 + m]^{s_2}_q[\delta_2 - m + 1]^{1-t_2}_q \mu_{\delta_2,m-1}. \tag{11b}$$

Equation (11*b*) is equivalent to (11*a*) provided that (10) takes place and neither of the q-numbers vanishes. If this is the case,

$$a_{m+1}=\frac{[\delta_2-m]_q^{t_2}[\delta_2+m+1]_q^{1-s_2}}{[\delta_1-m]_q^{t_1}[\delta_1+m+1]_q^{1-s_1}}\,\mu_{\delta_1,m}\mu_{\delta_2,m}a_m. \tag{12}$$

Clearly, the existence of non-zero bounded solutions for $\{a_n\}$ of (12) depends on the behaviour of $\mu_{\delta_i,m}$, $i=1, 2$, for large $|m|$. In general, different choices for $\{\mu_{\delta,m}\}$ lead to inequivalent representations. From now on we will restrict our attention to the particular case when all $\mu_{\delta,m}=1$.

Thus, we shall be concerned with infinite-dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ characterized by the following relations:

$$\begin{aligned} He_m &= me_m\\ X^-e_m &= [\delta+m]_q^s[\delta+m+1]_q^{1-t}e_{m-1}\\ X^+e_m &= [\delta-m]_q^t[\delta+m+1]_q^{1-s}e_{m+1}\\ Ce_m &= [\delta]_q[\delta+1]_qe_m. \end{aligned} \tag{13}$$

Our main tasks will be to find out which of these representations are equivalent and to answer the questions of reducibility. Both of them are connected with the possible vanishing of the q-numbers $[\delta\pm m]_q$, $[\delta\pm m+1]_q$. For $|q|\neq 1$ and $x\in\mathbb{R}$, $[x]_q=0$ is equivalent to $x=0$. If, however, $q=\exp(2\pi\mathrm{i}\beta)$ is a pure phase, (infinitely many) other solutions appear—one has then $[x]_q=\sin(\pi\beta x)/\sin(\pi\beta)$, and $[x]_q=0$ implies just $\beta x\in\mathbb{Z}$. When $\delta-\varepsilon\in\mathbb{Z}$, the relevant q-numbers can only vanish for rational β $(=p'/p$, where p' and p are coprime integers); such values of q are called non-generic. Hence, we have to consider separately the following four cases:

$$\begin{aligned} &\text{Case I.} && \delta-\varepsilon\notin\mathbb{Z}, |q|\neq 1\\ &\text{Case II.} && \delta-\varepsilon\notin\mathbb{Z}, |q|=1\\ &\text{Case III.} && \delta-\varepsilon\in\mathbb{Z}, q \text{ generic}\\ &\text{Case IV.} && \delta-\varepsilon\in\mathbb{Z}, q \text{ non-generic} \end{aligned} \tag{14}$$

Case I. The representation (13) is irreducible and neither a highest nor lowest weight one. Bounded non-zero solutions of (12) (with $\mu_{\delta_i,m}=1$, $i=1,2$) exist iff there are positive constants c_1, c_2 such that for any P, $Q\in\mathbb{N}-\varepsilon$

$$0<c_1<\prod_{m=-P}^{Q}\left|\frac{[\delta_2-m]_q^{t_2}[\delta_2+m+1]_q^{1-s_2}}{[\delta_1-m]_q^{t_1}[\delta_1+m+1]_q^{1-s_1}}\right|<c_2. \tag{15}$$

As (for any γ, $r\in\mathbb{R}$)

$$[\gamma+n]_q^r \underset{|n|\to\infty}{\propto} |q|^{\pm 1/2|nr|} \qquad (+ \text{ for } |q|>1,\ - \text{ for } |q|<1) \tag{16}$$

and due to the fact that (10) translates in this case into

$$(\delta_1-\delta_2)(\delta_1+\delta_2+1)=0 \tag{17}$$

one has to examine the two possibilities $\delta_2=\delta_1$ or $\delta_2=-\delta_1-1$. For example, when $\delta_2=\delta_1=\delta$,

$$[\delta-m]_q^{t_2-t_1}[\delta+m+1]_q^{s_1-s_2} \underset{|m|\to\infty}{\propto} |q|^{\pm 1/2|m|(|t_1-t_2|+|s_1-s_2|)} \tag{18}$$

and two representations $[\delta, \varepsilon; s_i, t_i]$, $i=1, 2$, are equivalent if and only if $s_1 = s_2$ and $t_1 = t_2$. For $\delta_2 = -\delta_1 - 1$ the relevant quantity in (15) is

$$[\delta_1 - m]_q^{1+s_2-t_1}[\delta_1 + m + 1]_q^{s_1+t_2-1} \underset{|m|\to\infty}{\propto} |q|^{\pm 1/2|m|(|1-s_2-t_1|+|1-s_1-t_2|)} \tag{19}$$

so that

$$[\delta, \varepsilon; s, t] \simeq [-\delta - 1, \varepsilon; 1-t, 1-s]. \tag{20}$$

One can therefore restrict δ to $\delta \geqslant -\frac{1}{2}$.

Case II. Let $q = \exp(2\pi i\beta)$, $0 < \beta < 1$ (s and t have to be restricted to $0 \leqslant s, t \leqslant 1$ so that the exponents in (13) are non-negative). The vector e_m would be a highest weight one if

$$m = \delta - k/\beta \qquad \text{or} \qquad m = -\delta - 1 - k/\beta \qquad \text{for some } k \in \mathbb{Z}. \tag{21a}$$

The corresponding lowest weight conditions read

$$m = -\delta - k/\beta \qquad \text{or} \qquad m = \delta + 1 - k/\beta \qquad \text{for some } k \in \mathbb{Z}. \tag{21b}$$

It is clear that (for $0 < s, t < 1$) either none of equations (21) has a solution or all of them have. In the first case (case IIa) the representation is irreducible. In the second case (case IIb) the solution is unique for β irrational and the representation splits into three parts—a highest-weight representation π_- with h.w. vector e_{-j-1} (for some $j \in \mathbb{N} - \varepsilon$), a finite-dimensional one, π_0, carried by the linear span of $\{e_{-j}, e_{-j+1}, \ldots, e_j\}$ and a lowest-weight representation π_+ with l.w. vector e_{j+1}. A degenerate case appears for $\varepsilon = 1/2$, $j = -1/2$ (there is no π_0 then). Let now $\beta = p'/p$ is rational (i.e. q is non-generic). It may happen again that equations (21) have no solution, hence this possibility falls into case IIa. But this time (only for some rational values of δ) each of the conditions (21) may be fulfilled for infinitely many (equally spaced) values of $m = N + kp$, $k \in \mathbb{Z}$ (case IIc).

We can consider now the various equivalence conditions. Note that for $q = \exp(2\pi i\beta)$, $0 < \beta < 1$, equation (10) is satisfied by

$$\delta_1 + \delta_2 = N/\beta \tag{22a}$$

or

$$\delta_1 + \delta_2 + 1 = M/\beta \tag{22b}$$

$N, M \in \mathbb{Z}$, so that

$$[\delta_1 - m]_q = [\delta_2 - m]_q \qquad [\delta_1 + m + 1]_q = [\delta_2 + m + 1]_q \tag{23a}$$

or, respectively,

$$[\delta_1 + m + 1]_q = -[\delta_2 - m]_q \qquad [\delta_1 - m]_q = -[\delta_2 + m + 1]_q. \tag{23b}$$

Case IIa. If equations (21) have no solution and β is irrational, then (15) cannot be satisfied uniformly in P, Q; this follows e.g. from the well known topological fact that the set of points $\{\exp(2\pi i m\beta)\}_{m\in\mathbb{Z}}$ with β irrational is dense in $\mathbb{S}^1$ so that some of the q-numbers (effectively, sinuses) would become arbitrarily small. Thus, for β irrational and any $N, M \in \mathbb{Z}$

$$[\delta + N/\beta, \varepsilon; s, t] = [\delta, \varepsilon; s, t] \simeq [-\delta - 1 + M/\beta, \varepsilon; 1-t, 1-s]. \tag{24}$$

For rational β (and δ) one needs other arguments.

Case IIb. Let e.g. $\delta_1 = M_1 + k_1/\beta$ (see (21)). Then e_{M_1} is a h.w. vector. The equivalence condition (22*a*) $\delta_2 = \delta_1 - N/\beta$, $N \in \mathbb{Z}$, leads to $\delta_2 = M_1 + (k_1 - N)/\beta$, i.e. e_{M_1} is a h.w. vector for the second representation too. If, on the other hand, (22*b*) is satisfied, then $\delta_2 = -M_1 - 1 + (M - k_1)/\beta$, i.e. e_{-M_1-1} is a h.w. vector also for the second representation. One immediately shows also that e_{M_1+1} and e_{-M_1} are l.w. vectors for both representations. So we have to consider equations (11) (with $\mu_{\delta_i,m} = 1, i = 1, 2$) instead of (12). The intertwining operators can be found up to overall multiplicative constants, a freedom intrinsic for the problem. Using the estimate (16) one shows that

$$\begin{aligned}\pi_\pm[\delta + K/\beta, \varepsilon; s, t|\pm(j+1)] \\ &\simeq \pi_\pm[\delta, \varepsilon; s, t|\pm(j+1)] \\ &\simeq \pi_\pm[-\delta - 1 + M/\beta, \varepsilon; 1-t, 1-s|\pm(j+1)] \end{aligned} \tag{25}$$

where the notation is enlarged to include the corresponding highest or lowest weights.

All finite-dimensional subrepresentations π_0 are equivalent to those with standard s and t (i.e. $s = t = 1/2$ in (13)).

One has to add to this list the cases when s and/or t take the values 0 or 1. Let e.g. $s = 0$, $t \neq 0, 1$. From (13), (21) one sees that a l.w. vector (e_{-j}, say) disappears from the general structure. One obtains an indecomposable h.w. representation $(\pi_- \oplus \pi_0) \supset \pi_0)$ for $j \geqslant 0$. If $t = 0$ too, e_j is no longer a h.w. vector and there are just two infinite-dimensional subrepresentations (π_+ and π_-) whose direct sum, however, does not exhaust the whole representation space etc.

Case IIc. The representation space splits (for $0 < s, t < 1$) into an infinite collection of finite-dimensional subrepresentations. Two infinite series of h.w. and l.w. vectors appear at m-values $(-j-1+rp, -j+rp)$, resp. $(j+lp, j+1+lp)$ where $r, l \in \mathbb{Z}$ and $m = j$ satisfies the first of (21*b*) for some $k \in \mathbb{Z}$. All finite-dimensional representations are again equivalent to those with $s = t = 1/2$. It may happen that a vector e_m be at the same time a h.w. and l.w. one (e.g. $e_{3/2}$ for $\delta = \frac{17}{6}$, $\varepsilon = \frac{1}{2}$, $s = t = \frac{1}{2}$, $\beta = \frac{3}{4}$). When s and/or t are equal to 0 or 1, the picture becomes different (as in the previously considered case IIb)—some of the h.w. or l.w. vectors disappear and some of the subrepresentations become indecomposable and possibly infinite dimensional.

Case III. As in case IIb, when $0 < s, t < 1$, there are just three irreducible representations, π_+ and π_-, both infinite-dimensional, and the finite-dimensional π_0. The equivalence conditions for $\pi_\pm$ read

$$\pi_\pm[j, \varepsilon; s, t|\pm(j+1)] \simeq \pi_\pm[-j-1, \varepsilon; 1-t, 1-s|\pm(j+1)]. \tag{26}$$

For $s = 0$, 1 and/or $t = 0$, 1 there is one or no h.w. or l.w. vector into the big reprensentation space. All finite-dimensional representations are again equivalent to those with $s = t = \frac{1}{2}$.

Case IV. The analysis of case IIc applies. For $0 < s, t < 1$ all representations are equivalent to those with $s = t = \frac{1}{2}$. Let $\delta = j \in \mathbb{Z} - \varepsilon$; then h.w. vectors are placed at $m = j + kp$ and $m = \tilde{j} + kp$ where $\tilde{j} = p - j - 1$ and $k \in \mathbb{Z}$, and l.w. vectors—at $m = -j + kp$ and $m = -\tilde{j} + kp$. Therefore the two sets of representations $[j + kp, \varepsilon; \frac{1}{2}, \frac{1}{2}]$ and $[\tilde{j} + kp, \varepsilon; \frac{1}{2}, \frac{1}{2}]$, k, $1 \in \mathbb{Z}$ have the same scheme of h.w. and l.w. vectors. As is well known, the q-generalizations of the usual spin-j representations are not always irreducible in this case.

We end with a remark on the existence of invariant Hermitian forms. Let B be a bounded operator for which

$$(e_n, BX^{\mp}e_m) = (X^{\pm}e_n, Be_m) \qquad (e_n, BHe_m) = (He_n, Be_m). \tag{27}$$

The last equality means that $B_{nm} = (e_n, Be_m)$ obey

$$(m-n)B_{nm} = 0 \qquad \text{i.e. } B_{nm} = b_n\delta_{nm}. \tag{28}$$

The first two equalities are then reduced to

$$b_{n+1} = [\delta+n+1]_q^{2t-1}[\delta-n]_q^{1-2s}b_n. \tag{29}$$

In case I ($\delta - \varepsilon \notin \mathbb{Z}$, $|q| \neq 1$) bounded solutions of (29) only exist for $s = t = \frac{1}{2}$.

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